Wittgenstein’s Philosophy of Mathematics – A Re-Assessment Starting from the Critique of Cantor’s Proof of the Uncountability of the Real Numbers
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0 Introduction and Content

Wittgenstein’s notes on mathematics are fragmentary, but nevertheless precise and coherent — this is Christine Redecker’s position with regards to the topic of her book. Starting from Wittgenstein’s critique of Cantor’s diagonal proof she promises to give a reassessment of his philosophy of mathematics. She considers this critique to be neither as groundless as his opponents hold, nor as harmless as some of his defenders present it.

In the latter part of the book Redecker highlights some constructivistic, conventionalistic and revisionistic elements in Wittgenstein’s philosophy of mathematics.

According to the table of content the author considers her book as divided into two parts (”Wittgensteins Critique of Cantor’s Diagonal Proof in RFM II, 1-22”, and ”Wittgenstein’s Critique in the Context of His Philosophy of Mathematics”), but at least for the purpose of this review it seems more appropriate to split it into three parts: the first dealing in detail with Cantor’s diagonal argument, the second concerned with the broader topic of real numbers, and the third discussing the most general question of Wittgenstein being some kind of -ist (anti-Platonist, constructivist, intuitionist, finitist,...).

There are interesting discussions in the book I omit: detailed comparisons with paradoxes and back-references to Cantor’s diagonal argument throughout the later part book.

As the book is written in German I spend much more effort than usual on just reproducing its content. Nevertheless I hope that my presentation might also be useful for people knowing the book, as I figure out the main lines of argumentation, which might not be so easy to grasp in the case of this book (as Redecker confronts every claim with several anticipated objections, so that the content of the book becomes very dense and full up with details).
1 Wittgenstein’s Critique of Cantor’s Diagonal Argument

According to Redecker, Wittgenstein’s critique concerns five points (see p. 29f):

(1) whether the diagonal sequence defines a $b$-adic fraction and hence a real number (RFM II, 1-8)
(2) whether the diagonal number is different from all elements of the list (RFM II, 9-11)
(3) whether the diagonal proof really is a proof (RFM II, 12-15)
(4) whether there is a clear concept of countability (RFM II, 16-21)
(5) whether the diagonal procedure proves that there are infinite sets of different cardinality (RFM II, 22)

In a nutshell, her arguments are as follows.

1.1 Does the diagonal sequence define a real number?

If someone says: “Shew me a number different from all these”, and is given the diagonal rule for answer, why should he not say: “But I didn’t mean it like that!”? What you have given me is a rule for a step-by-step construction of numbers that are different from each of these successively. (RFM II, 3)

Redecker takes this and the following paragraphs of RFM as aiming at two points:
Wittgenstein wants to show, first, that the diagonal number in Cantor’s proof cannot be defined in any other way than by the diagonal procedure; it has therefore, to use Wittgenstein’s terminology, no “surrounding” (RFM II, 126). Redecker explains by comparing two examples: if you build a suitable diagonal number for the list of square roots of natural numbers, you can decide that this diagonal number is not a member of the list in a finite number of steps, so you do not need the whole diagonal procedure for the proof. The situation is different if you consider a list of the rational numbers in $[0, 1]$, because then (as the rational numbers lie dense in $\mathbb{R}$) it might not be possible to see in any other way than by using the whole diagonal procedure that the number is not an element of the list. The same is the case in Cantor’s proof. This means that you cannot cope with a diagonal number of a list of the real numbers in $[0, 1]$ in any other way than by using the diagonal procedure, it is not determined by any properties besides the one “being the result of the (a certain) diagonal procedure”. Hence the meaning of such a diagonal number depends only on the diagonal procedure (and in general Wittgenstein hesitates to call something which has no surrounding a “meaning”).

This gives much weight to the fact that, second, the diagonal argument does not “by itself” generate a real number (how could it?), but only if we presuppose (some properties of) the real numbers (for example, the axiom of completeness of $\mathbb{R}$ or the theorem that every Cauchy-sequence converges to a real number).

On top of this comes the fact that Wittgenstein does not accept that every arbitrarily built decimal expansion defines a real number, he only accepts this in certain cases, for example if the coefficients of the expansion are defined recursively. (In Chapter Five of her book Redecker argues this in detail.)
Much later (p. 132f) Redecker comes back to this topic. She examines what a situation might look like in which we would accept the diagonal procedure as creating by itself a real number. If we had learnt a finite diagonal procedure to generate new numbers from childhood on, we would consider it a natural way of gaining numbers and hence use it as a matter of course even in the infinite case. We would use it as we now use the axiom of completeness.

1.2 Is the diagonal number different from all elements of the list?

Suppose it were now said: the development of the diagonal series never catches up with the other series:—certainly the diagonal series avoids each of those series when it encounters it, but that is no help to it, as the development of the other series is again ahead of it. Here I can surely say: There is always one of the series for which it is not determined whether or not it is different from the diagonal series. It may be said: they run after one another to infinity, but the original series is always ahead. (RFM II, 9)

Redecker wants to explain why it is important which of the numbers “is ahead” and to make Wittgenstein’s argument defensible even in a mathematically more precise version. To do so, one thing is crucial: to argue how it could be possible, in the usual framework of mathematics, for the diagonal number not to be different from all the numbers of the infinite list (because only then Wittgenstein’s dictum “there is always one of the series for which it is not determined whether...” can make sense).

Let $D_\infty$ be the diagonal number, $V_n$ the $n$-th element of the list. Redecker distinguishes between “different from every number of the list” and “different from all numbers in the list”. Instead of the usual understanding of “different from all numbers in the list” $\forall n \in \mathbb{N} \ (D_\infty \neq V_n)$ she proposes to define $\forall n \in \mathbb{N} \ (D_\infty \neq V_n) \iff \exists \varepsilon > 0 \ (|D_\infty - V_n| > \varepsilon)$.

The first equivalence expresses the Cantorian view; the second expresses the Wittgensteinian view as Redecker interprets it (p. 86). That means that it would be a matter of definition whether it is correct to say that the diagonal number is different from all numbers of the list (whereas it is clear that it is different from every number of the list). (I will comment on this in the last section of this review.)

The “Wittgensteinian definition” justifies what Redecker states earlier (p. 58) in her book. She holds that Wittgenstein in the quoted passage of paragraph 9 opposes $\forall n \in \mathbb{N} \exists k \in \mathbb{N} \ (V_k = D_n) \iff \forall k \in \mathbb{N} \ (V_k \neq D_\infty)$.

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1Wittgenstein uses “series” rather colloquially, actually meaning “sequence” –as it was still not unusual at that time– in most cases.
There is always one of the series for which it is not determined whether or not it is different from the diagonal series to Cantor’s

$$\forall n \in \mathbb{N} \forall k \in \mathbb{N} \,(k \leq n \rightarrow V_k \neq D_n) \rightarrow \forall k \in \mathbb{N} \,(V_k \neq D_\infty),$$

where $D_n$’s decimal expansion is equal to that of $D_\infty$ up to the $n$-th decimal place and is 0 thereafter.

The antecedents are compatible, but a contradiction between the consequents can only be avoided by understanding $\forall k(V_k \neq D_\infty)$ not in the usual way, but according to the Wittgensteinian definition.

1.3 The diagonal ”proof”

Redecker discusses whether the diagonal ”proof” is indeed a proof, a paradox or the definition of a concept. Her considerations first return to the problem of understanding “different from an infinite set of numbers” in an appropriate way, as the finite case does not fix the infinite case.

It means nothing to say: “Therefore the X numbers are not denumerable”. One might say something like this: I call number-concept X non-denumerable if it has been stipulated that, whatever numbers falling under this concept you arrange in a series, the diagonal number of this series is also to fall under the concept. (RFM II, 10)

What does the diagonal ”proof” really prove? At the end of the proof one faces the following situation: the diagonal number should by presupposition be contained in the list, but the arguments of the proof show that it cannot be contained in the list. Redecker comments, the diagonal argument pretends to show that the list [Aufzählung] produces a well-defined number, which is not contained in the list. But it only shows that a number contained in the list, which ought to be the diagonal number of the list, cannot be completely defined. (p. 112f, transl. E. R.)

For if it is not defined completely by the diagonal argument, there is no contradiction. The impossibility of the equality of the diagonal number with any number of the list therefore presupposes that the diagonal number is different from all numbers in the list\(^2\) (it must be presupposed as being completely defined and different from all numbers of the list, otherwise there is no contradiction). Therefore the diagonal proof is not a proof, but a(n implicit) determination of concepts (see my summary below).

1.4 Countability

The mistake begins when one says that the cardinal numbers can be ordered in a series. For what concept has one of this ordering? (RFM II, 16)

\(^2\)Note that Redecker again uses her or rather Wittgenstein’s definition for “different from all numbers”.

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Cardinal numbers are, as Redecker remarks, finite cardinals for Wittgenstein here. There are a variety of possible meanings of “ordering a set (N) in a series”. For example, it could mean that a ordering relation or a well-ordering is defined on it. Therefore it is not clear, what ”ordered in a series” means. But within the diagonal proof it seems clear that it is countability only that matters. Redecker objects:

The diagonal proof does not immediately revert to the concept of countability, but makes use of the fact that sets, which can be mapped one-to-one onto the natural numbers, can be “ordered in a series” according to the succession of the natural numbers by means of this mapping. (p. 125, transl. E. R.)

1.5 Infinite sets of different cardinality

The difference between the real numbers and the natural numbers, which becomes evident by the diagonal proof, is not a difference between the cardinalities of two sets, as it is not independent of the ordering (“Anordnung”) of the sets. (p. 141)

Redecker defines a "δ-series" to be a sequence with ordinal number ω having the property that the (however defined in particular) diagonal number is a real number. And she argues that it is "being a δ-series", that Wittgenstein means by "ordering in a series" in connection with Cantor’s diagonal argument. But from the fact that the real numbers cannot be “ordered in a series” in this sense it does not follow that the set of real numbers has larger cardinality than the set of natural numbers, nothing is said about the “number of objects”. Moreover, Richard’s paradox shows that cardinality is not a suitable means for determining the magnitude of a set: Suppose the set V of all real numbers, which can be described by finitely many letters, to be arranged in alphabetical order of these descriptions. Consider the number which is described by “the number whose n-th decimal place differs (in a specified way) from that of the n-th number in the list for all n”. By definition this number cannot be an element of the list; but on the other hand it is clearly described by an expression consisting of finitely many letters and therefore must be an element of this list. This contradiction is usually solved by stating that V is not well-defined. But there is, according to Redecker, another, more plausible, way out: If we stick to the claim that the set is well-defined, then the consequence is that there are subsets of countable sets which cannot be arranged in a δ-series. V cannot be mapped one-to-one to N, as such a function could never be surjective as the diagonal number is never contained in the list of the numbers defined before. Hence, there are infinite subsets of N, which are not countable. This disqualifies countability as a means for determining the magnitude of a set (as a subset should not be “larger”). As a consequence, the usual concept of countability should be replaced by ”can be ordered in a (δ-)series”.

1.6 Summing up

According to Redecker the diagonal “proof” is a determination of a concept in three respects (p.131). First, the sequence of values resulting from a diagonal procedure gives something which is called a "real number". Second, this number is said to be “different from the list of real numbers”. Third, the “proof” defines “countability” (as “being ordered in a sequence of ordinal number ω”).
2 Wittgenstein and the real numbers

Redecker describes Wittgenstein's view on the real numbers as an on-going development from the *Tractatus* (the "operational view") to the RFM. The early Wittgenstein presents almost a small theory on the real numbers, whereas in his later writings there are only scattered remarks. In *Philosophical Remarks* (PR 181) Wittgenstein states that "the irrational number is not the extension of an infinite decimal fraction, but a law". Redecker gives a detailed discussion of "extension" and argues that the concept of "law" and the change of its meaning is crucial for Wittgenstein's attitude against the real numbers all along. The principle of verification, the criterion of comparability, and the principle of equality of types are involved in these considerations, but in the end comparability becomes the key criterion. Redecker holds that Wittgenstein develops this opinion in PG and does not change it anymore in RFM. A formulation and justification of the criterion of comparability is given in PR:

> It seems to be a good rule that what I will call a number is that which can be compared with any rational number taken at random. (PR, 191)

The determination of real numbers as laws rules out the usual definitions by axioms, Cauchy-sequences or Dedekind-cuts.

3 Wittgenstein and some -isms

3.1 Constructivism and Platonism

Redecker opposes Platonism and constructivism (in a wide sense) –in such a way that it becomes a question of either-or to be a constructivist or a Platonist, so that who is not a platonist, must be a constructivist and vice versa– and states that Wittgenstein is beyond doubt a constructivist in the sense that he thinks that mathematical objects are created by the work of mathematicians (RFM I, 168). Things become more difficult with respect to more sophisticated specifications of Platonism and constructivism. Several versions of constructivism are discussed in the book. A main feature of intuitionism is the non-acceptance of the tertium non datur for mathematical propositions. Redecker describes the debate of Shanker, Fogelin, and Marion, especially on the question whether it is undecidable propositions only, for which the tertium non datur is invalid. She comes to the conclusion that, first, Wittgenstein’s scepticism towards the tertium non datur is not primarily connected with undecidable propositions, and, second, logical laws cannot be applied to mathematical propositions, as mathematical propositions are normative, not descriptive (and the opposite of a prescript is not again a prescript) (cf. p. 259).

Strict finitism holds that mathematical expressions are meaningful only if they refer to constructions, which can de facto be executed. This definition is used in works of Dummett, Esenin-Volpin, Marion, Wright, and others. Wittgenstein is sometimes said to be a strict finitist, and this is motivated by his emphasis on surveyability. But his understanding of "surveyable" differs from that of the strict finitists. They would argue that 67\((257^{729})\) is not a number, as it is by definition equal to 1 + 1 + 1 + ..., with 67\((257^{729})\) ones added, which is not surveyable and the calculation cannot be executed. Wittgenstein, by contrast, holds that
$6^{257^{729}}$ is surveyable (although nobody could ever count this far), for exponentiation is something different, understanding it means following different rules (than adding ones).

3.2 Conventionalism

Redecker ascribes to Michael Dummett one of the most influential formulations of what is called “conventionalism” (p. 282). Conventionalism is characterised by the view that (mathematical) necessity is imposed by us not on reality, but upon language. The “full-blooded” conventionalism differs from a “modified” conventionalism insofar as the first sees every mathematical proposition as the expression of a convention, which can be accepted or denied, whereas the second does so only for the axioms and basic rules, from which the other mathematical propositions necessarily follow. According to Dummett, Wittgenstein is a full-blooded conventionalist in his sense. Redecker disagrees: neither is it the case that only conventions of language determine a mathematical proposition, for it may also be implicit rules we have learned to follow; nor is it true, that we may accept or refuse every step of a mathematical proof and every mathematical proposition ad libitum. On the contrary, if there were general or large-scale disagreement over the result of a calculation, for example, we would not call it “calculation” anymore (RFM III, 73). Redecker refers (p. 289) to Barry Stroud, who points out that it is a central property of mathematical inferences and operations that not any arbitrary result will be accepted.

Nevertheless there are elements of Wittgenstein’s philosophy of mathematics that might be called “conventionalistic”, although these conventions are not arbitrary, but based on practice. Redecker:

> The necessity of a mathematical proposition results from the utility of the corresponding techniques of calculations for our daily life, which made it indispensable to use rules of calculations accordingly and not to doubt them. (p. 303, transl. E. R.)

The chapter ends with an explanation of how Wittgenstein unifies his conventionalistic and verificationistic tendencies.

3.3 Revisionism

In the last chapter of the book Redecker aims at providing a revisionistic re-interpretation of Wittgenstein’s philosophy of mathematics. She argues that the anti-revisionistic readings of Maddy and Wright as well as other, more moderate anti-revisionistic readings are incoherent. Wittgenstein’s critique of Cantor is an example of a consideration which would indeed, if taken seriously, change something within mathematics. The passages, which are usually quoted as stating an anti-revisionistic standpoint (PI 124, RFM II, 62, RFM III, 31), should be understood, according to Redecker, as a “division of work” between the mathematician and the philosopher, the first developing theories, the second studying the established systems and calculations.
4 Objections and Conclusion

The part of the book dedicated to Cantor’s diagonal argument is beyond doubt one of the most elaborated and precise discussions of this topic. Although Wittgenstein is often criticized for dealing only with elementary arithmetics and this topic would be a chance for Wittgenstein scholars to show that he also made interesting philosophical contributions to what can (with some good will) be called "higher mathematics", his remarks on Cantor are widely neglected. Redecker offers a useful and important work and starting point for thoughts about Cantor’s proof, even if one cannot be convinced that Wittgenstein himself studied the material in such (technical) detail as it is done in Redecker’s book.

The following critical remarks are not meant to say that Redecker is wrong on any specific point, but just to indicate various directions in which further considerations might lead. My first remark is just a supplement to section 1.1 of this review. If you consider the diagonal argument in a formally precise version, then what is needed to identify the diagonal number as a real number is only a quite uncontroversially accepted part of mathematics. Suppose \( \varphi \) were a 1-1-mapping from \( \mathbb{N} \) to \([0,1]\). For each \( b \in [0,1] \) there is a decimal notation, let’s say 0, \( b_1 b_2 b_3 \ldots \). Define functions \( g_i : [0,1] \to [0,1] \) such that \( g_i(b) = 0,0\ldots 0 b_i00\ldots \) Then the series \( \sum_{n=1}^{\infty} (g_n(\varphi(n)) + 10^{-n}) \) converges (for \( N \) tends to infinity) and therefore has as its limit a real number \( d \), let’s say (between 0 and 1). It can easily be shown that \( d \) is different from all numbers \( \varphi(n) \), \( n = 1,2,\ldots \). This is a contradiction to the assumption that \( \varphi \) is one-to-one. If someone claims this proof to be false, especially \( d \) not to be a real number, the most plausible targets for his attack will be the (countable) axiom of choice and the convergence of Cauchy-sequences. Wittgenstein does not mention the axiom of choice in this context, therefore we have to focus on the (countable) axiom guaranteeing the convergence of Cauchy-sequences. Perhaps Wittgenstein indeed wanted to deny this, but then one has to be aware of the enormous consequences of such a denial – most of the main theorems of Analysis (the Calculus) would cease to hold, theorems, which have proved themselves in practice, even in the practice of engineers.

A concern of a similar kind applies to Redecker’s interpretation I described in section 1.2. Her determination of “different from all elements of a list” is either an additional definition besides the usual one, then it has no place and no task in mathematics yet, and it would have to turn out in future time how it could become useful. Or it is meant as a definition substituting the usual one, then it has to be said, that the usual Analysis would obviously completely break down. This would be hard-core revisionistic.

Redecker’s discussion of the status of Cantor’s proof, of which I gave a very brief sketch in section 1.3, is very sophisticated, and my comment is just coming “from outside”, just addressing the outcome of her argumentation: that a not completely defined entity (rather than the limit of a sequence of rationals) should be a number seems hard to swallow to me. I think from the remarks I have made thus far my major concern should have become clear: to change mathematical concepts in the context of one mathematical theorem without regarding the consequences for mathematics as a whole does not go well with an emphasis on mathematics’ “utility […] for our daily life”.

The discussion concerning the concept of cardinality (section 1.4 and 1.5) relies on the assumption that Wittgenstein had an understanding of “ordering in a list”, which does not
coincide with “countability” in the usual meaning. But neither does such an alternative concept of “ordering” matter for Cantor’s proof (Redecker herself mentions this possible objection, but nevertheless insists on a different understanding of “ordering in a list”), nor is there, as far as I see, any hint that Wittgenstein wanted to use ”denumerable” unconventionally. (I suggest the following alternative interpretation of the passages she is dealing with: the proof starts with “suppose the real numbers between 0 and 1 to be ordered in a list” or “suppose there is a bijection between N and...”. I take Wittgenstein to argue that this is a nonsensical presupposition. We do not understand what “a list of XY” or a “bijection between N and...”, where XY is not countable, should mean.)

The second and the third part of Redecker’s book deserves nothing but commendation. Even if one is sceptic against the discussion of -isms in connection with Wittgenstein in general, Redecker’s way of handling it is very appealing and useful. She explains, specifies, and clarifies the positions to such an extent that, what is actually at issue, is not a ”position” anymore, but an important and clear-cut tendency of Wittgenstein’s philosophy of mathematics.

References


Wittgenstein (second from right), summer 1920. v. t. e. Ludwig Wittgenstein considered his chief contribution to be in the philosophy of mathematics, a topic to which he devoted much of his work between 1929 and 1944. As with his philosophy of language, Wittgenstein's views on mathematics evolved from the period of the Tractatus Logico-Philosophicus: with him changing from logicism (which was endorsed by his mentor Bertrand Russell) towards a general anti-foundationalism and constructivism that was Between these two peaks, Wittgenstein famously retreated from philosophy and for a time decamped to the role of a school teacher. During this spell he produced a 42-page pronunciation and spelling dictionary for children, the only book of his apart from the Tractatus that was published in his lifetime.Â So this is my imperfect and doubtless incomplete description of the philosophy of Ludvig Wittgenstein. I think of it more of a method, really, a means of looking beyond the tendency that runs deep within our culture to look for permanent and essential definitions of the entities around us. As an art historian, Iâ€™ve relied upon Wittgensteinâ€™s method many times as arguments over â€œis it art?â€ rage about me. Frege, Russell and Wittgenstein have had a unique and powerful influence on almost all aspects of twentieth century analytic philosophy. A study of these authors is thus an excellent introduction to a good range of the most important contemporary debates in philosophy. Study in this area requires that you should know the work of at least two of these authors (somewhat artificially, Wittgenstein's early and late work are counted as separate bodies of work for this requirement). Â The False Prison: a Study of the Development of Wittgenstein's Philosophy. 2 Vols. Oxford: Clarendon Press. Cambridge Core - Twenty-first-Century Philosophy - Wittgenstein's Philosophical Investigations. Wittgenstein's philosophy of mathematics. In Truth and Other Enigmas. London: Duckworth. Review of Wolfgang Kienzler, Wittgensteins Wende zu seiner Spätphilosophie 1930â€“1932. European Journal of Philosophy 6: 379â€“85. Schulte, J. 2002. Discover Book Depository's huge selection of Christine Redecker books online. Free delivery worldwide on over 20 million titles. Wittgensteins Philosophie der Mathematik. Christine Redecker. 01 Sep 2006. Hardback.