Risk Measures for Derivative Securities: From a Yin-Yang Approach to Aerospace Space

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A Brief History of Binomial Tree

- Yin-Yang: I-Ching or Zhouyi (1,000 BC or before) and Taoism (the late 4th century BC)

- Origin in Probability Theory: Daniel Bernoulli (29 January 1700 - 17 March 1782); Coin tossing experiment \{H, T\}

- Discrete-time binomial tree in finance: Bill Sharpe?

- A beautiful paper by Cox, Ross and Rubinstein, CRR, (1979): Option valuation in a discrete-time binomial model
Behind the scene: Boyle, Siu and Yang (2002)

- Asian Financial Crisis in 1997: LTCM and derivative securities

- Reappraisal of Value at Risk (VaR): Non-Subadditivity

- Coherent risk measures by Artzner, Delbean, Eber and Heath (1999)

- Tail risk, expected shortfall and a research report in Bank of Japan by Yamai and Yoshiba (2002)
The Challenge

- Traditional theories in finance: Linear risk

- Capital Asset Pricing Model and Arbitrage Pricing Theory

- Bigger universe of nonlinear risk: not well-explored!

- Examples: Derivative securities and hedged funds

- Current Practice: Traders use Greek Letters, such as Delta, Gamma, Rho, ..., etc.

- Consider a discrete-time financial model consisting of a risk-free bond $B$ and a stock $S$

- Deal with a European call option $C$ written on $S$ with strike price $K$ and maturity $T$

- Build the two-level binomial model from the CRR binomial model

- Evaluate a coherent risk measure, namely Expected Shortfall (ES), for derivative securities
The Model

- Suppose \( \{0, 1, 2, \ldots, T\} \) is the time parameter set in the first level.

- For each time point \( k \) in the first level, \([k, k + 1]\) is the time interval for risk measurement.

- Divide \([k, k + 1]\) into \( m \) equal sub-intervals.

- Then \( \{0, 1, 2, \ldots, km, km + 1, \ldots, Tm\} \) is the time parameter set in the second level.
• For each sub-interval \([n, n + 1]\) in the second level, assume that, under a real-world probability measure \(\mathcal{P}\),
\[
\frac{B_{n+1}}{B_n} = \hat{r}
\]
\[
\frac{S_{n+1}}{S_n} = \begin{cases} 
  u & \text{with probability } p \\
  d & \text{with probability } 1 - p
\end{cases}
\]

• Call price from the CRR binomial model:
\[
C_{km} = \frac{1}{\hat{r}^{T_m-k_m}} \sum_{j=0}^{T_m-k_m} \binom{T_m-k_m}{j} q^j(1-q)^{T_m-k_m-j} (S_{km}u^j d^{T_m-k_m-j} - K)^+
\]
Expected Shortfall (ES) for the Call

- $\Delta C_{k,m}$: the discounted net loss $C_{km} - \hat{r}^{-m}C_{(k+1)m}$ of the call option $C$ over $[km, (k + 1)m]$

- $F_{km}$: the information generated by the values of $S$ up to and including time $km$

- Under $\mathcal{P}$, the distribution of $\Delta C_{k,m} | F_{km}$:

$$\Delta C_{k,m} = C_{km} - \hat{r}^{-m}C_{(k+1)m}(S_{km}u^j d^{m-j})$$

with probability $\binom{m}{j} p^j (1 - p)^{m-j}$, $j = 0, 1, \ldots, m$. 
• **ES for the call $C$:**

\[
ES_\alpha(\Delta C_{k,m} | \mathcal{F}_{km}) = E_P(\Delta C_{k,m} I_{\{\Delta C_{k,m} \geq \text{VaR}_\alpha\}}, \mathcal{F}_{km}) = \alpha^{-1}[E_P(\Delta C_{k,m} I_{\{\Delta C_{k,m} \geq \text{VaR}_\alpha, P(\Delta C_{k,m} | \mathcal{F}_{km} ) \}} \mathcal{F}_{km}) + \text{VaR}_\alpha, P(\Delta C_{k,m} | \mathcal{F}_{km}) (\alpha - P(\Delta C_{k,m} \geq \text{VaR}_\alpha, P(\Delta C_{k,m} | \mathcal{F}_{km} ) | \mathcal{F}_{km})]
\]

• **Adjustment for the discrete loss distribution to ensure the coherent property for the ES**
An Expression for the ES

- Define \( j_\alpha = \sup\{j \in J \mid \Delta C_{k,m}(j) \geq \text{VaR}_{\alpha, P}(\Delta C_{k,m}|F_{km})\} \), where \( J \) represents the set \( \{0, 1, 2, \ldots, m\} \). Then

\[
ES_\alpha(\Delta C_{k,m}|F_{km}) = -\frac{1}{\hat{\sigma}_{Tm-km}^\alpha} \left\{ \sum_{j=0}^{j_\alpha} \binom{m}{j} p^j (1 - p)^{m-j} \left[ \sum_{i=0}^{Tm-(k+1)m} \binom{Tm-(k+1)m}{i} \left( Tm - (k+1)m \right)^i \right] q^i (1 - q)^{Tm-(k+1)m-i} (S_{km} u^i d^{Tm-km-j-i} - K)^+ \right\} - \sum_{i=0}^{Tm-km} \binom{Tm-km}{i} (Tm - km)^i q^i (1 - q)^{Tm-km-i} (S_{km} u^i d^{Tm-km-i} - K)^+ \]
Numerical Example

• Consider a European call with $T = 2$ months and $K = 22$. Suppose $S_0 = 25$

• Assume that the time horizon for measuring the risk of the position is one month

• $r = 0.7\%$ per month and $\sigma = 6\%$ per month

• Two-level binomial model: $m = 5$, $u = e^{0.0268}$, $d = e^{-0.0268}$ and $q = 0.5194$
The numerical values of ES and VaR for the call

Table: ES and VaR for various values of $p$ and $\alpha$.

<table>
<thead>
<tr>
<th>$p \setminus \alpha$</th>
<th>0.01</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>$2.795653$ ($2.795653$)</td>
<td>$2.795653$ ($2.795653$)</td>
</tr>
<tr>
<td>0.4</td>
<td>$2.795653$ ($2.795653$)</td>
<td>$2.795653$ ($2.795653$)</td>
</tr>
<tr>
<td>0.5</td>
<td>$2.795653$ ($2.795653$)</td>
<td>$2.49328$ ($1.989324$)</td>
</tr>
<tr>
<td>0.6</td>
<td>$2.795653$ ($2.795653$)</td>
<td>$2.15446$ ($1.989324$)</td>
</tr>
<tr>
<td>0.7</td>
<td>$2.185262$ ($1.989324$)</td>
<td>$1.591582$ ($0.852669$)</td>
</tr>
</tbody>
</table>
Yang-Yin Grows Everything

- Yang-Yin generates many patterns

- Think about the modern computing technologies

- Central Limit Theorem: Binomial $\Rightarrow$ Normal

- CRR binomial model $\Rightarrow$ Continuous-time Black-Scholes-Merton model
Risk Measures for Derivatives in Continuous-Time Markets


- “Bang-Bang” type control: Use in Aerospace engineering

- Paul Wilmott’s book on Quantitative Finance and uncertain volatility models widely used in the finance industry
Risk Measures in Elliott, Siu and Chan (2008)

- Consider a financial model consisting of a bank account $B$ and a share $S$

- A continuous-time, $N$-state observable Markov chain $\{X(t)\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\{e_1, e_2, \ldots, e_N\}$.

- The price dynamics for $B$ and $S$ under $\mathcal{P}$:

  $$dB(t) = rB(t)dt,$$
  $$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t),$$

  where $\mu(t) := \langle \mu, X(t) \rangle$ and $\sigma(t) := \langle \sigma, X(t) \rangle$; $\mu := (\mu_1, \mu_2, \ldots, \mu_N)'$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)'$. 
First Step: Valuation


- The regime-switching Esscher transform by Elliott, Chan and Siu (2005):

  1. Define a process \( \theta := \{\theta(t)\} \) by: \( \theta(t) = \langle \theta, X(t) \rangle \), where \( \theta = (\theta_1, \theta_2, \ldots, \theta_N)' \).

  2. The regime-switching Esscher transform \( Q_\theta \sim P \) associated with \( \theta := \{\theta(t)\} \):

\[
\frac{dQ_\theta}{dP}\bigg|_{G(t)} := \frac{\exp(\int_0^t \theta(u)dW(u))}{E[\exp(\int_0^t \theta(u)dW(u))|F^X(t)]}.
\]
• Consider a European-style option with payoff \( V(S(T)) \) at maturity \( T \).

• Given \( S(t) = s \) and \( X(t) = x \), a conditional price of the option is given by:

\[
V(t, s, x) = \mathbb{E}^{\theta}[e^{-r(T-t)}V(S(T))|S(t) = s, X(t) = x].
\]

• **Proposition 1:** Let \( V_i := V(t, s, e_i) \), for each \( i = 1, 2, \cdots, N \), and write \( V := (V_1, V_2, \cdots, V_N)' \in \mathbb{R}^N \). Write \( A(t) \) for the rate matrix of the chain at time \( t \). Then, \( V_i, i = 1, 2, \cdots, N \), satisfy the following system of \( N \)-coupled P.D.E.s:

\[
-rV_i + \frac{\partial V_i}{\partial t} + rs \frac{\partial V_i}{\partial s} + \frac{1}{2} \sigma_i^2 s \frac{\partial^2 V_i}{\partial s^2} + \langle V, A(t)e_i \rangle = 0,
\]

with terminal conditions \( V(T, s, e_i) = V(S(T)), i = 1, 2, \cdots, N \).
Second Step: Risk Evaluation

• For each $i = 1, 2, \cdots, N$, let $\Lambda_i = [\lambda_i^-, \lambda_i^+]$. For example, when $N = 2$ (i.e. State 1 is “Good Economy” and State 2 is “Bad Economy”), $\lambda_1^- = 0.05$; $\lambda_1^+ = 0.10$; $\lambda_2^- = 0.01$; $\lambda_2^+ = 0.05$.

• Suppose $\lambda(t)$ is the subjective appreciation rate of the share at time $t$. The chain modulates $\lambda(t)$ as:

\[
\lambda(t) = \langle \lambda, X(t) \rangle ,
\]

where $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_N)' \in \mathbb{R}^N$ with $\lambda_i \in \Lambda_i$, $i = 1, 2, \cdots, N$. 
• Consider, for each $\lambda \in \Theta$, a process $\{\theta^\lambda(t)\}$ defined by putting

$$\theta^\lambda(t) = \sum_{i=1}^{N} \left( \frac{\mu_i - \lambda_i}{\sigma_i} \right) \langle X(t), e_i \rangle .$$

• The regime-switching Esscher transform $\mathcal{P}_{\theta\lambda} \sim \mathcal{P}$ on $\mathcal{G}(t)$ with respect to $\{\theta^\lambda(t)\}$:

$$\left. \frac{d\mathcal{P}_{\theta\lambda}}{d\mathcal{P}} \right|_{\mathcal{G}(t)} := \frac{\exp(\int_0^t \theta^\lambda(u) dW(u))}{\mathbb{E}[\exp(\int_0^t \theta^\lambda(u) dW(u))|F^X(t)]} .$$

• Under $\mathcal{P}_{\theta\lambda}$,

$$dS(t) = \lambda(t)S(t)dt + \sigma(t)S(t)dW^\lambda(t) ,$$

where $\{W^\lambda(t)\}$ is a $(\mathcal{G}, \mathcal{P}_{\theta\lambda})$-standard Brownian motion.
• Future net loss of the option position over \([t, t + h]\):

\[
\Delta V(t, h) := e^{rh}V(t, S(t), X(t)) - V(t + h, S(t + h), X(t + h))
\]

• Given \(S(u) = s\) and \(X(u) = x\), \(u \in [t, t + h]\), the generalized scenario expectation for the option position \(V\) over \([t, t + h]\):

\[
\rho(u, s, x)
\]

\[
:= \sup_{\lambda \in \Theta} \mathbb{E}^{\theta_{\lambda}}[\exp(-r(t + h - u))\Delta V(t, h)|S(u) = s, X(u) = x],
\]

where \(\mathbb{E}^{\theta_{\lambda}}[\cdot]\) is an expectation under \(\mathcal{P}_{\theta_{\lambda}}\).

• Write \(\rho_i := \rho(u, s, e_i), i = 1, 2, \cdots, N\), and \(\rho := (\rho_1, \rho_2, \cdots, \rho_N)'\).
• Proposition 2. For each \( i = 1, 2, \ldots, N \), let \( \Delta_i^R := \frac{\partial \rho_i}{\partial s} \) and 
\[
\lambda(\Delta_i^R) = \begin{cases} 
\lambda_i^+ & \text{if } \Delta_i^R > 0 \\
\lambda_i^- & \text{if } \Delta_i^R < 0 
\end{cases}
\]
Then \( \rho_i, i = 1, 2, \ldots, N \), satisfy the following system of \( N \)-coupled P.D.E.s:

\[
\frac{\partial \rho_i}{\partial u} + \frac{1}{2} \sigma_i^2 s^2 \frac{\partial^2 \rho_i}{\partial s^2} + \lambda(\Delta_i^R) s \frac{\partial \rho_i}{\partial s} - r \rho_i + \langle \rho, A(t) e_i \rangle = 0 ,
\]
with the following terminal conditions:

\[
\rho(t+h, S(t+h), e_i) = e^{rh} V(t, S(t), X(t)) - V(t+h, S(t+h), e_i) .
\]

• For the case of an American-style option, a system of coupled variational inequalities for the risk measures was obtained.
What Next?

- Incorporate credit risk and counterparty risk in the OTC markets

- Liquidity risk due to large trading positions

- Applications to modern insurance products with embedded options

References


Space activities have followed a near-identical phasing in most of the space-faring nations: - Political objectives are set by governments with direct/indirect impacts on the space sector (e.g. US military navigation GPS, human spaceflight programs for prestige); - Those objectives are executed by governmental/space agencies, together with the related funding. Agencies tend to define the related projects and programs, which are executed under contract by the national space industry; - The space industry develops a number of products and processes and may urge for the commercialization of some FAA Aerospace Forecasts. Appendix A: Alternative Forecast Scenarios. Appendix B: FAA Forecast Accuracy. Looking forward, there is confidence that U.S. airlines have finally transformed from a capital intensive, highly cyclical industry to an industry that generates solid returns on capital and sustained profits. Fundamentally, over the medium and long term, aviation demand is driven by economic activity, and a growing U.S. and world economy provides the basis for aviation to grow over the long run. The 2019 FAA forecast calls for U.S. carrier domestic passenger growth over the next 20 years to average 1.8 percent per year. Abstract of "The standardised approach for measuring counterparty credit risk exposures - final document", March 2014 The Basel Committee's final standard on "The standardised approach for measuring counterparty credit risk exposures" includes a comprehensive, non-modelled approach for measuring counterparty credit risk associated with OTC derivatives, exchange-traded derivatives, and long settlement transactions. The new standardised approach (SA-CCR) replaces both the Current Exposure Method (CEM) and the Standardised Method (SM) in the capital adequacy framework. Increased specificity regarding the application of the approach to complex instruments; the introduction of a supervisory measure of duration for interest rate and credit derivative exposures.